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Potential flow along a wavy wall and transonic controversy

G. BOLLMANN

Grunewaldstr. 81, 1000 Berlin 62, West Germany

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SUMMARY

The flow of an inviscid, compressible, perfect gas along a sinus-shaped wall is used as a model to shed light into the long-standing transonic controversy. The solution of the small-disturbance approximation for the velocity potential is developed as a formal series in the similarity parameter k. Forty terms in k were obtained, delegating the computational work to a computer. The coefficients of the maximal-speed series turn out to be moments with positive weight on a finite interval of support. This implies

$$u_{max} = \sum_{n=0}^{\infty} a_n k^n = \int_a^b \frac{dw\,(x)}{1-kx}\ .$$

The solution of the classical moment problem, i.e. the recovering of the weight distribution, shows that $-a = b = 1/k_c$, with k_c as the critical value of the parameter, at which the flow first becomes sonic. k_c has been determined as 0.8253, the four digits being regarded as definite. It follows that u_{max} as a function of k is analytic on $(-k_c, k_c)$ and has exactly two singularities located on the real axis at $k = -k_c$ and $k = k_c$. It is known from the theory of moments that there does not exist an analytic continuation on the real axis exceeding the interval. This means that the velocity depends analytically on the parameter as long as the local velocity of sound is nowhere reached, but that the exceeding of it is marked by termination of analyticity and, to this degree, is critical. Thus, the view that there will not exist a transonic potential flow having neighbour solutions is supported. The smallness of the weight distribution at the interval ends (at least at $k = -k_c$ it is even exponentially small, giving an exponentially small singularity of u_{max}) is the key to explaining, within the scope of the inviscid model, the shockless exceeding of the critical value by some per cent seen in numerical calculations and experiments.

1 The transonic controversy

About 50 years ago G.I. Taylor [1] raised the question whether an observed breakdown of flow near the critical Mach number occurs because irrotational flow ceases to be possible or occurs for some other reason connected with viscosity or other properties of the air not considered in the theory. Realising experimentally by electrical analogy an iteration procedure similar to that of Rayleigh-Janzen, he observed a breakdown of convergence when the free-stream Mach number exceeded the critical one by 3–4%, likewise for flow about a circular cylinder and that past an airfoil. A local calculation giving the same result convinced him that the failure of convergence was not due to the method of successive approximations but was indeed caused by non-existence of solutions. Concerning the accuracy of the approximations and experiments, it remained open whether the critical Mach number might be exceeded by some per cent or not at all.

During the following years some explicit solutions to the inviscid, compressible-flow equation were found. These solutions were analytical functions of the co-ordinates and described shock-free flow with local supersonic enclosures [2–6]. They gave rise to the opinion that the local velocity of sound was not critical for smooth potential flow. But Busemann [7, 8], Guderley [9] and Frankl [10] published arguments supporting the view that these transonic flows were exceptional, having no shock-free neighbours if the physical data, i.e. the free-stream Mach number or the contour, were slightly changed and therefore were without physical significance. Indeed, they did not allow the physical parameters to vary independently, because they were completely constructed by the indirect hodograph method. The boundary is not known in the hodograph (velocity plane); only particular solutions of the (linear) equation can be found, and the profile has to be realised afterwards as a streamline. The hodograph method was also used by C.S. Morawetz, who proved in a series of papers [11] conjectures on the non-existence of smooth transonic flow. But her proofs did not match the full requirements on the admissible class of distortions, especially on the smoothness of a contour variation, and thus could not decide the transonic controversy, a term first used by L. Bers [12]. A critique is found in the book by Ferrari and Tricomi [13]. After all these attempts to decide the controversy using the hodograph method, Tricomi expressed the opinion that this approach would never succeed and that one would deal better with the non-linear equation in the physical plane [14].

In contrast to theoretical arguments, there are several experiments and numerical investigations that seem to contradict the non-existence conjectures; Koppe's impressive experiments [15] and the numerical results of Nocilla et al. [16] should be mentioned. In connection with 'practical' shock-free transonic airfoils constructed within the last decade, it was recently stated by Sobieczky et al. [17] that their numerical results imply the existence of infinitely many 'numerical shockless' neighbour configurations, a statement that does not necessarily contradict the conjectures, which claim that the boundary-value problem is not well posed. But the controversy is too delicate a problem – as will be confirmed here – to admit a solution by discretisation methods for the mixed non-linear equation. Questioning the idealised inviscid model may seem to offer an easy explanation of the encountered discrepancies, but the model itself is probably just insuffiently explored by mathematical analysis. L. Bers reasoned that: 'Experience shows that in most cases the value of an idealized model can not be assessed a priori. Only after the mathematics of the model has been sufficiently explored can one tell under which circumstances the model gives a satisfactory description of physical reality and in which cases it fails' [18].

Indeed, the following analysis will show that the discrepancy between theoretical arguments for non-existence of smooth transonic flow and the mentioned observations could be explained within the scope of the inviscid model.

2. Parameter expansion

As already pointed out, the non-existence conjectures contend that the mixed boundary-value problem is ill posed: its solution cannot depend continuously on the free-stream Mach number or on boundary data. Therefore, it seems promising to look for a solution as an expansion into one of these parameters. For the flow about a cylinder the M^2-expansion, or Rayleigh-Janzen, method has extensively been used; recently Van Dyke and Guttmann got 29 terms in M^2 for a circular cylinder [19]. For the flow along thin airfoils, an expansion into a thickness parameter is suitable and offers in addition the use of a transonic approximation. The first term of the M^2-expansion gives the incompressible case; for the thickness-parameter expansion it is identical with the Prandl-Glauert rule. Both methods produce successive approximations as solutions of purely elliptical equations, and it has therefore been doubted [12] that the limit could be a solution to the mixed elliptic-hyperbolic problem. But even if that were to be substantiated, there could be an analytical continuation over the critical value of the parameter. The analyticity in the parameter – not merely the convergence – is the real subject of an analysis of the series.

In 1978 Van Dyke and Guttmann analysed 19 terms of the maximum-speed series for the circular cylinder and came to the conclusion that it was convergent at about 4% above the critical Mach number [20]. But the analysis of the scheme of extrapolations for the radius of convergence showed a strange drifting not in accordance with conditions it was based on. Milton Van Dyke suggested to me that I look at the thickness-ratio expansion for the flow along a wavy wall. Indeed, the corresponding series revealed exactly the same behaviour. Analysing recently 29 terms of the M^2- series, Van Dyke and Guttmann [19] claim now that it is convergent at 1.1% above the critical Mach number. But they could not extract the nature of the singularity limiting the convergence.

3. Transonic flow past a sine-shaped wall

The flow past a wavy wall, a simple model due to the periodicity in one co-ordinate has been the object of several theoretical and experimental investigations [21, 22, 23]. In spite of the simplicity of its sine shape, no explicit solution is known. This simple model seems to be well suited for argumentation within the transonic controversy, since a variation of the thickness parameter (or equivalently the amplitude) is an analytical deformation of the geometry. Further, it allows one to go over to the transonic small-disturbance approximation bearing the name of von Karman [24]. This is considered exact for a double limit in the free-stream Mach number and the thickness ratio throughout the flow field (there are no stagnation points). This approximation has been chosen, since the computational complexity is reduced. For a subsonic free-stream Mach number ($M_\infty < 1$) with the y-coordinate scaled by $(1 - M_\infty^2)^{1/2}$ the small-disturbance approximation can be formulated as

$$(1 - k\phi_x)\phi_{xx} + \phi_{yy} = 0. \tag{1}$$

Here $k = \tau(\gamma + 1)(1 - M_\infty^2)^{-3/2}$ is the transonic similarity parameter (usually $k^{-2/3}$ is defined

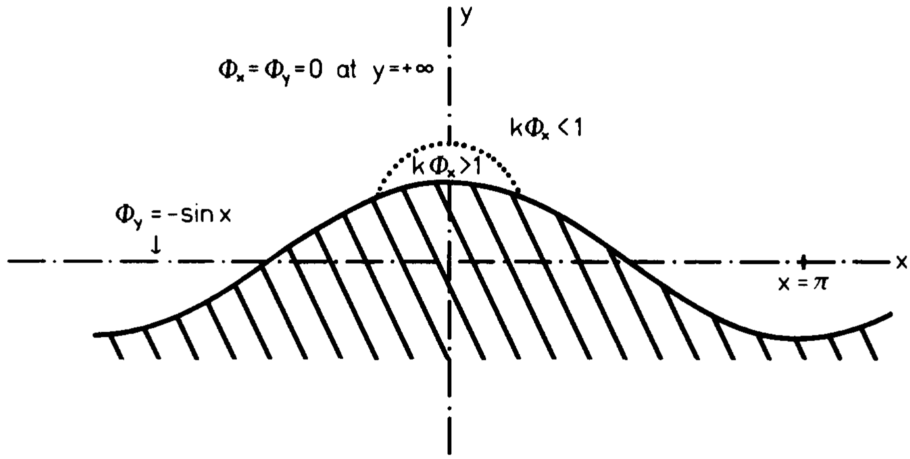

Figure 1 Boundary conditions.

as similarity parameter, admitting also $M_\infty > 1$) $\tau = 2\pi$ amplitude/wavelength = the thickness parameter, M_∞ = free stream Mach number and γ = adiabatic ratio.

Equation (1) is elliptic for small k, $1 - k\phi_x > 0$, and hyperbolic for $1 - k\phi_x < 0$. The equation $1 - k\phi_x = 0$ defines the sonic line.

As long as the equation is elliptic at every point, it can be assumed that there exists a unique solution which is analytical in the co-ordinates and in the parameter k; the maximum of the velocity will be reached at the peak of the curve, even though there seems to be no proof covering exactly this configuration. For the mixed problem, where the equation is allowed to be locally hyperbolic a uniqueness theorem was proved for the flow about a thin airfoil [25].

The critical value k_c of the parameter k at which the flow first becomes locally sonic is defined as

$$1 - k_c\phi_x\,(0, 0, k_c) = 0. \tag{2}$$

The boundary condition can be imposed at $y = 0$, as has been justified in [26]. The transonic small-disturbance equation admits solutions with local supersonic flow containing weak shocks [27, 28]. Moreover, it is a local model near the sonic line [29] and an exact equation for a hypothetical gas ('Tricomi Gas') [12]. In general it is not known if the solution of the full equation will converge in the corresponding limit to the solution of the approximate equation, but for the case of the wavy wall it is obviously true [22]. To get the parameter expansion, equation (1) is written as

$$\Delta\phi = k\phi_x\phi_{xx}, \tag{1a}$$

and the successive iterations are

$$\Delta\phi^{(1)} = 0, \tag{3}$$

$$\Delta\phi^{(2)} = k\phi_x^{(1)}\phi_{xx}^{(1)},$$

$$\Delta\phi^{(N)} = k\phi_x^{(N-1)}\phi_{xx}^{(N-1)},$$

where for convenience only the next order in k has to be kept at each step. The formal solution

$$\phi(x,y,k) = \sum a_{jmln} k^j e^{-my} y^l \sin(nx) \tag{4}$$

was calculated by Kaplan [23] up to $N = 8$ and turned out to be without errors. Kaplan's coefficients served to debug a fortran program that gave 40 terms in k. This result might be considered optimal because all three limiting factors came together at this point: execution time was 25 minutes in IBM 3033 and is proportional to N^8; storage was nearly one million bytes and $\sim N^4$; quadruple precision was used equivalent to about 34 decimal places – the loss of accuracy allowing one to get only a few more terms. The loss of digits was determined by runs with fewer decimal places on other computers (CDC6400, Cray-1). The following analysis is restricted to the maximum-speed series $u_{max} = u_{max}(k) = u(0, 0, k)$. By the same series the velocity at the trough is computed using

$$u(\pi, 0, k) = -u(0, 0, -k). \tag{5}$$

The coefficients for u_{max} are given in Table 1.

4. Analysis of the maximum-speed series

The critical value k_c is not known in advance and is calculated from

$$\sum_{i=1}^{n} a_i k^i(n) = 1 \tag{6}$$

approximating equation (2). Since $a_i > 0$, every polynomial equation has exactly one positive solution $k(n)$. The $k(n)$ decrease monotonically for $n = 1,2 \ldots$ and form a sequence converging to k_c if the maximum speed series has a radius of convergence $k_0 \geqslant k_c$. Otherwise an extrapolation would lead to an erroneous result. The first case is assumed, since the differential equation is elliptical for $k < k_c$ and, as will be shown, the analyticity terminates at $k = k_0$. Looking at the convergence of $k(n)$ from (6) for different right-hand sides instead of 1 confirms the assumption. The critical value was determined as

$$k_c = 0.8253, \tag{7}$$

the four digits being considered as definite.

The analysis of the power series – or, more precisely, the function represented by the series

Table 1. Coefficients for max. speed series

0.100000000000000000	D + 01
0.500000000000000000	D + 00
0.104166666666666674	D + 01
0.234027777777777768	D + 01
0.701909722222222232	D + 01
0.207975115740740755	D + 02
0.706670235339506156	D + 02
0.236367861689814813	D + 03
0.857721065297067923	D + 03
0.307233174527883159	D + 04
0.116196819850933525	D + 05
0.435026855823935539	D + 05
0.169328334031523860	D + 06
0.653896820555007493	D + 06
0.259999587798802811	D + 07
0.102738903781264347	D + 08
0.415300699140180051	D + 08
0.167048413967045370	D + 09
0.684235204449248433	D + 09
0.279153035242287731	D + 10
0.115588671763974657	D + 11
0.477076708505482578	D + 11
0.199346261517804764	D + 12
0.830775133387831696	D + 12
0.349837314729274268	D + 13
0.146996483551611265	D + 14
0.623152374979021094	D + 14
0.263695266434290305	D + 15
0.112441832142358219	D + 16
0.478746995691134400	D + 16
0.205197418333933500	D + 17
0.878411717817077760	D + 17
0.378231989859059104	D + 18
0.162692579126353178	D + 19
0.703423169365990349	D + 19
0.303870726559031214	D + 20
0.131872101296252355	D + 21
0.571876297249282458	D + 21
0.249019354482231129	D + 22
0.108368009663787513	D + 23

– has to be done in the complex k-plane, for the convergence might be limited by a non-real singularity. In the paper by Gaunt and Guttmann [30] an account of the most used methods is given. If the nearest singularity is of algebraico-logarithmic type, as often encountered in fluid mechanics and critical phenomena problems, then the Domb-Sykes plot, a_n/a_{n-1} vs. $1/n$, will approach a straight line; the singularity is easily extrapolated and higher polynomial interpolations in $1/n$ (Neville table) will settle down. The other widely used method is the Padé approximation which gives a picture of the location of singularities through its poles and zeros.

Not much is known about convergence of Padé approximation except for the Stieltjes and Hamburger cases [33, 34]. Padé approximations for u_{max} (k) gave the simple picture of only two singularities located on the real axis, a positive and a negative one, both of modulus close to the critical value k_c. This provided an explanation for the waviness in the Domb-Sykes plot and offered a remedy: to map away one singularity by an Euler transformation or to treat even and odd terms of the series separately. Either case led to visually straight plots and an extrapolated radius of convergence $k_0 = k_c + 3\%$. But the Neville table showed a slight continuous drift towards the critical value, the gap became smaller and smaller. It was exactly the same strange behaviour reported by Van Dyke and Guttmann [19] in the case of the M^2-expansion for the circle. Like them, I did not succeed in explaining it by means reported in [30], taking logarithmic derivatives, assuming confluent singularities, etc.

Key to a successful analysis was the discovery that the coefficients were not only positive but proved to be moments with positive weight $dw(x)$:

$$a_n = \int_a^b x^n dw\ (x) \quad (=) \quad \int^b x^n \rho\,(x)\,dx \tag{8}$$

(the first integral is understood in Stieltjes's sense, the second representation is valid under additional conditions for $w(x)$).
The power series can be represented then as

$$u_{\max}(k) = u(0,0,k) = \int_a^b \frac{dw\,(x)}{1-xk} \quad (=) \quad \int_a^b \frac{\rho(x)\,dx}{1-xk} \tag{9}$$

The moment property is valid, iff the Hankel determinants are positive:

$$a_0 > 0, \begin{vmatrix} a_0 & a_1 \\ a_1 & a_2 \end{vmatrix} > 0, \qquad \begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix} = 0, \ldots \tag{10}$$

Naturally, this has been verified only for the finite number of coefficients of the power series, but the tendency raises no doubts that the positiveness will continue. The determinants proved to be very sensitive to round-off errors, so that an accuracy of the coefficients to ten digits was required, as discovered by a run on another computer admitting slightly fewer digits.

The moment problem, i.e. recovering the weight distribution from the moments, has a long tradition and is closely connected with some other fundamental areas of mathematics. Many results are found in Akhieser's book [31]. The Gauss quadrature formula with a distribution $dw(x)$ on the interval can be constructed from the moments alone [32] and is for an integrand $1/(1-kx)$ identical with the $(N-1/N)$ Padé approximation for the moment series. They are convergent in this case [33, 34]. The corresponding orthogonal polynomials satisfy recursion formulars, whose coefficients allow conclusions on the behaviour of the weight function [35]. Outside the interval of orthogonality, under recently weakened conditions [36], asymptotic behaviour of the quotients of these polynomials has been proved. The moment problem is also connected with the theory of continued fractions. Some of these relations have been used for the analysis of $u_{\max}$; they confirm the results given below or are at least not in contradiction to them. In principle all of the information is contained in the discrete approximation for $\rho\,(x)$

that was calculated from α_i (β_i) (the discrete approximation for $dw(x)$) by dividing through the distribution of the zeros β_i, approximately given by $\beta_i - \beta_{i-1}$ vs. $(\beta_i + \beta_{i-1})/2$. The approximation for $\rho(x)$ is shown in Figure 2. The α_i and β_i were determined as corresponding weights and knots of the Gauss quadrature formula according to the method given in [32]. At the same time $1/\beta_i$ is a pole and α_i/β_i is the corresponding residue of the $(N-1/N)$ Padé approximation to the power series.

Figure 2 shows clearly that there is an exponentially small singularity at $k = -k_c$, whereas the behaviour near $k = k_c$ remains uncertain. The analysis carried out separately for the even and odd parts of the power series did not clarify the situation completely but supported the assumption that the interval of support will be $(-1/k_c, 1/k_c)$, since the corresponding weight distributions were nearly coincident close to the point in question. The analysis was possible because both turned out to be moment series too, with a positive interval of support, thus being of the well-known Stieltjes type. If the interval of support is the same for the even and for the odd parts, then the interval for the entire series is symmetrical near zero. There is a second more convincing argument. The velocity at the bottom of the trough is given by the maximum-speed series through (5) and therefore also as a moment series with weight distribution $-\rho(-x)$. From the moment representation it follows that there does not exist an analytical continuation over the singularities given by the reciprocals of the interval ends, as known from the theory of moments. If the interval of support does not reach $1/k_c$, then the maximum speed would continue to be analytical for k exceeding the critical value. At the same value there would be a break-down at the bottom of the trough, where the differential equation is still elliptical. Therefore it will be assumed that $dw(x) > 0$ on $(-1/k_c, 1/k_c)$:

$$u_{\max} = u(0,0,k) = \sum_{n=0}^{\infty} a_n k^n = \int_{-1/k_c}^{1/k_c} \frac{dw(x)}{1-kx} \; (=) \int_{-1/k_c}^{1/k_c} \frac{\rho(x)dx}{1-kx} \; . \qquad (11)$$

5. Discussion and conclusion

From the representation of the maximum-speed series as an integral with positive weight distribution, it follows that the maximum speed is an analytic function of the similarity parameter k, as long as nowhere in the flow field the local velocity of sound is reached. This analyticity ceases to exist when the flow first becomes sonic. The break-down of analyticity in k seems to be a global feature in contrast to analyticity in the co-ordinates. This does not imply directly that the mixed boundary-value problem is not well posed, since that would require unsteadiness in k, but shows that exceeding the local velocity of sound is critical.

It will be conjectured that the extraordinary structure of the maximum function will hold for other geometries or parameter expansions too, since the maximum speed, being subsonic, is a monotone function of the free-stream Mach number, at least for symmetrical profiles [12]. This conjecture is supported by similar behaviour for the M^2-expansion for the circle concerning its positiveness and for the drifting of the Neville table reported by Van Dyke and Guttmann [19].

The disagreements between theoretical arguments for non-existence of smooth transonic

Table 2. Zeros and weights for corresponding Gauss quadrature formula

β_i	α_i
−4.30148469831711178	0.000000599231103220671489
−3.94726668724070429	0.0000174503075902077571
−3.49621086034580308	0.000277656360949791146
−2.95146969956890670	0.00117075363480424044
−2.32388384828863459	0.00412845781185060007
−1.62793554633976156	0.0109596936716737830
−0.877770177652766673	0.0231464912800360013
−0.0454166971703261445	0.0453128008193010223
0.313735784949430691	0.0704224426277678256
0.987734676980918200	0.107785006285090051
1.67774475639938259	0.131312324035862779
2.34284996903925236	0.158683359036653898
2.95563376349214613	0.151054080250720812
3.49445973903366003	0.137697346293580491
3.94231850876617895	0.0930433955894373332
4.28778772681974294	0.0514105261083624742
4.52977268589428506	0.0135776166552164347

$$\sum_{i=1}^{N} \frac{\alpha_i}{1-\beta_i k} \simeq \int_a^b \frac{dw(x)}{1-xk}$$

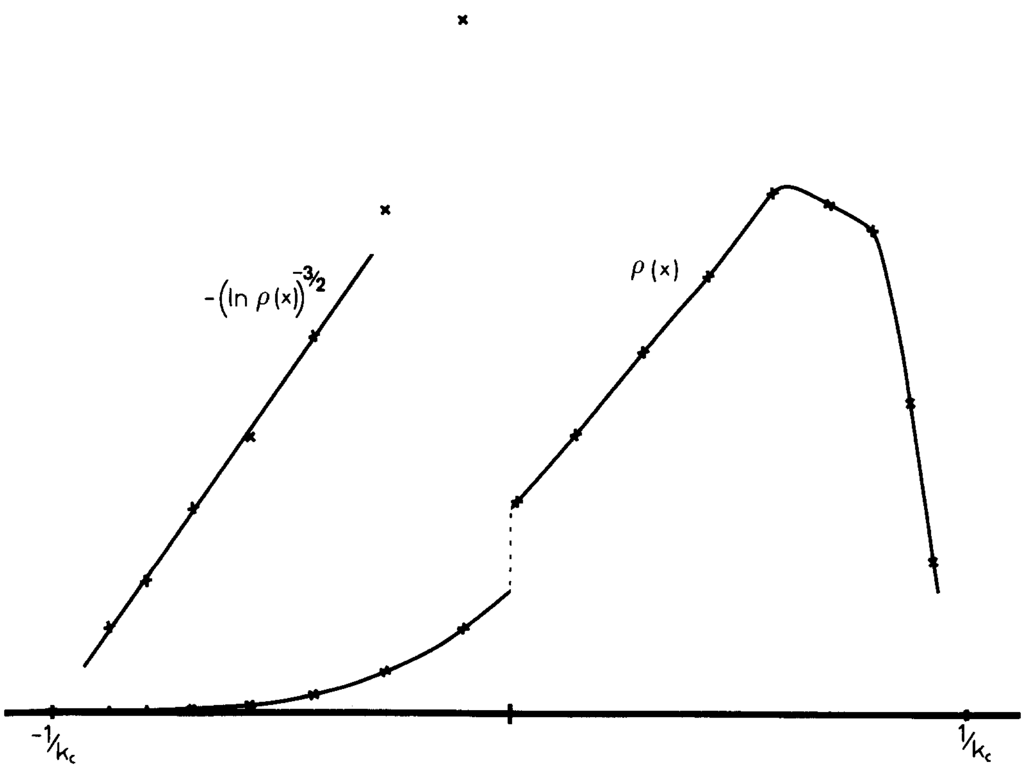

Figure 2. Weight function

flow, experiments and numerical results can be explained through an interpretation of the recovered weight distribution. It is seen from Figure 2 that the weight is very small at the interval ends, at least at one end even exponentially small, so that it seems unlikely that any discretisation method for the differential equation could approximate it correctly. But the cutting off of the distribution, leading to a smaller interval of support, would be equivalent to a greater range of convergence. Thus G.I. Taylor's results were already optimal, showing a convergence by some per cent above the critical Mach number. The extremely small singularities, cutting purely subsonic flow from a mixed subsonic-supersonic one, were difficult to detect and have led to erroneous results concerning the existence of transonic shock-free flow.

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